Tighter Analysis of an Approximation for the Cumulative VRP*

Mauro Henrique Mulati[1,2], **Flávio Keidi Miyazawa**[1]

[1]Institute of Computing – University of Campinas (UNICAMP)
Campinas – SP – Brazil

[2]Department of Computer Science – Midwestern State University (UNICENTRO)
Guarapuava – PR – Brazil

{mhmulati,fkm}@ic.unicamp.br

Abstract. *We deal with the cumulative vehicle routing problem (VRP), a generalization of the capacitated VRP, which objective is to minimize the fuel consumption. Gaur et al. in 2013 gave a 4-approximation based on a well-known partition heuristic to the traveling salesperson problem (TSP). We present a tighter analysis obtaining a* $\left(4 - \frac{4}{3s^3Q^2}\right)$*-approximation, where* Q *is the capacity of the vehicle and* s *is a scaling factor. To the best of our knowledge, this is the best proved approximation for the cumulative VRP so far.*

1. Introduction and Previous Work

The cumulative VRP was proposed by [Kara et al. 2008]. The objective is to minimize the fuel consumption, given that the fuel consumed by distance unit is linearly proportional to the total weight being carried (vehicle + load).

An instance of the cumulative VRP is given by what follows. A complete undirected graph $G(V, E)$ with vertices $V = \{0, 1, \ldots, n\}$, where 0 is the depot and the other vertices are customers. There is an object of weight $w_i \in \mathbb{Q}_{>0}$ for each customer i, and we consider that $w_0 = 0$. Each edge $uv \in E$ has a length $d_{uv} \in \mathbb{Q}_{>0}$ satisfying the triangular inequality. An empty vehicle with capacity $Q \in \mathbb{Q}_{>0}$ and weight $W_0 \in \mathbb{Q}_{>0}$ is initially located at the depot, and also, the weight of an object does not exceed Q. In a feasible solution S, we have that the only vehicle is repeatedly used in k directed cycles, each one including the depot, to form a schedule that picks up the objects at the customers and drops them in the depot, visiting every customer exactly once. The objective is to obtain such a schedule that minimizes the fuel consumed.

Let $\mu > 0$ be a constant that relates how much fuel is consumed by weight per distance unit. We define $a = \mu W_0$ and $b = \mu$. The fuel consumed by the vehicle to traverse the cycle C_j is $f(C_j) = a|C_j| + b\sum_{i \in C_j} w_i d_i^S$, where $|C_j|$ is the length of the cycle C_j, and d_i^S is the distance traveled by the vehicle carrying the object from being picked in the customer i until being dropped at the depot in the schedule S. The fuel consumed by a vehicle to perform a schedule S is $f(S) = a\sum_{j=1}^{k}|C_j| + b\sum_{i=1}^{n} w_i d_i^S$. Let d_i be the shortest distance between vertex i and the depot.

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In this paper, we roughly follow the structure of [Gaur et al. 2013]. For our contribution, we use the theorems 1-4 from [Gaur et al. 2013] and the definitions presented below.

Theorem 1 ([Haimovich and Rinnooy Kan 1985, Gaur et al. 2013])**.** *Let C^* denote an optimal TSP tour of the graph $G(V,E)$. Then, the total distance traveled by a vehicle to bring all objects to the depot is at least* $\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right)$.

A *subtour* is a TSP tour that visits a subset of $V(G)$, and, when it is clear, we will use tour to denote it. W.l.o.g., consider that the vertices of a tour C are numbered as $0, 1, \ldots, n$ in the order that they appear in the sequence, and 0 is the depot. $|C|$ is the length of the tour C. By a *cluster* $[i, j]$ we mean a set of a sequence of vertices in the tour C from i to j, with the extremes included. Considering $k \geq 2$, and $1 < x_1 < x_2 < \cdots < x_{k-1} \leq n$, a *cluster partition* denoted by $P = [1, x_1, x_2, \ldots, x_{k-1}, n]$ of tour C is a decomposition of C into the k clusters $[1, x_1]$, $[x_1, x_2], \ldots, [x_{k-1}, n]$. From a cluster partition P of C, we are able to construct k subtours $C_1, \ldots, C_k$ such that: to traverse the subtour C_j, the vehicle begins at the depot, visits the vertices of C_j in increasing order, and ends in the depot. The length of P is given by $l(P) = |C_1| + \cdots + |C_k|$.

Theorem 2 ([Altinkemer and Gavish 1987, Gaur et al. 2013])**.** *Let an integer $s > 0$ be a scaling factor in a way that sw_i, for every $i \in V$, sW_0, and sQ are positive integers, such that $sw_i \leq sQ$ for every $i \in V$. Let C be a TSP tour of G and let Q be the vehicle capacity. Then, there exists a cluster partition P of C with total length at most* $4\frac{\sum_{i=1}^{n} w_i d_i}{Q} + \left(1 - \frac{2}{sQ}\right)|C|$.

Theorem 3 ([Gaur et al. 2013])**.** *Let C^* be an optimal TSP tour, and let Q be the capacity of the vehicle. Then, the minimum fuel consumed by the vehicle to bring all objects to the depot is at least $a \cdot \max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + b\left(\sum_{i=1}^{n} w_i d_i\right)$.*

Theorem 4 ([Gaur et al. 2013])**.** *Let $\beta > 0$ be a positive rational number, C be a TSP tour, and assume that the vehicle has infinite capacity. Then, there exists a cluster partition $P = [1, x_1, x_2, \ldots, x_{k-1}, n]$ of C with total fuel consumption at most* $\left(1 + \frac{2}{\beta}\right) b\left(\sum_{i=1}^{n} w_i d_i\right) + \left(1 + \frac{\beta}{2}\right) a|C|$.

2. Our Contribution

We provide, in theorems 5 and 6, a refined analysis of the algorithm of [Gaur et al. 2013], showing a tighter approximation ratio than the one presented by them.

Theorem 5 (From [Gaur et al. 2013] with a tighter bound)**.** *Let $\beta > 0$ be a positive rational number, C be a TSP tour, and Q be the vehicle capacity. Then, there exists a cluster partition $P = [1, x_1, x_2, \ldots, x_{k-1}, n]$ of C with total fuel consumption at most* $\left(1 + \frac{2}{\beta}\right) b\left(\sum_{i=1}^{n} w_i d_i\right) + \left(1 + \frac{\beta}{2}\right) a|C| + 4a\frac{\sum_{i=1}^{n} w_i d_i}{Q} - 2a\frac{\sum_{j=1}^{k} |C_j|}{sQ}$.

Proof. Considering infinite capacity, there exists a cluster partition P of tour C with fuel consumption $f(P)$ with an upper bound given by Theorem 4. Let $C_1, C_2, \ldots, C_k$ be the subtours corresponding to the cluster partition P. Let W_j be the total weight of the objects picked by the vehicle in the subtour C_j. If $W_j \leq Q$, then C_j satisfies the capacity restriction and, consequently, we keep the cluster corresponding to the subtour C_j with fuel consumption $f(C_j)$ unchanged. On the other hand, assume that $W_j > Q$: by

Theorem 2 there exists a refined cluster partition P_j of C_j such that the total weight of the objects in each cluster of P_j is at most Q, and there exists an upper bound on its length.

Now, given a subtour C_j, we will give an upper bound on the fuel consumption $f(P_j)$. W.l.o.g., we assume that the fuel consumption $f(C_j)$ of the subtour C_j is obtained by a traversal in clockwise order (the reversed case is symmetric). Consider that the vehicle traverses each subtour $C_{jl}, 1 \leq l \leq k_j$ in the partition P_j. We have that V_j is the set of vertices of the tour C_j. Consider that, for each vertex $i \in V_j$, we have $d_i^{C_j}$ that represents the distance traveled by the vehicle from picking object i to dropping it at the depot in tour C_j, and analogously, $d_i^{P_j}$ represents the respective distance for the cluster partition P_j. We have that $d_i^{P_j} \leq d_i^{C_j}$, because the cluster partition P_j is a refinement of the tour C_j. Thus we can write $f(P_j) = \sum_{l=1}^{k_j} f(C_{jl}) = \sum_{l=1}^{k_j} \left(a|C_{jl}| + b \sum_{i \in C_{jl}} w_i d_i^{P_j} \right) = a \sum_{l=1}^{k_j} |C_{jl}| + b \sum_{i \in C_j} w_i d_i^{P_j} \leq a \cdot l(P_j) + b \sum_{i \in C_j} w_i d_i^{C_j} \leq 4a \frac{\sum_{i \in C_j} w_i d_i}{Q} + a|C_j| - a|C_j| \frac{2}{sQ} + b \sum_{i \in C_j} w_i d_i^{C_j} = f(C_j) + 4a \frac{\sum_{i \in C_j} w_i d_i}{Q} - a|C_j| \frac{2}{sQ}$, where we used the upper bound on $l(P_j)$ given by Theorem 2.

We consider that P' is the final cluster partition that includes all the clusters, the ones unchanged as well as the ones refined. Thus, the total fuel consumption is given by $f(P') = \sum_{j=1}^{k} f(P_j) \leq \sum_{j=1}^{k} \left(f(C_j) + 4a \frac{\sum_{i \in C_j} w_i d_i}{Q} - a|C_j| \frac{2}{sQ} \right) = \sum_{j=1}^{k} f(C_j) + 4a \frac{\sum_{j=1}^{k} \sum_{i \in C_j} w_i d_i}{Q} - 2a \frac{\sum_{j=1}^{k} |C_j|}{sQ} = f(P) + 4a \frac{\sum_{j=1}^{n} w_i d_i}{Q} - 2a \frac{\sum_{j=1}^{k} |C_j|}{sQ}$. By the upper bound of Theorem 4 on $f(P)$, we have that P' is a cluster partition satisfying the theorem. ■

Lemma 1. *Let C^* be an optimal TSP tour in a complete graph $G(V, E)$ with weight function d that are part of an instance of the capacitated VRP. Then, $|C^*| \leq 2 \sum_{i=1}^{n} d_i$.*

Proof. Recall that $|V| = n + 1$, with the vertices numbered from 0 (depot) to n. By definition $|C^*| = \sum_{uv \in E(C^*)} d_{uv} \leq \sum_{uv \in E(C^*)} (d_u + d_v) = \sum_{i \in V(C^*)} 2d_i = 2 \sum_{i=0}^{n} d_i = 2 \sum_{i=1}^{n} d_i$, where we used: that $d_{uv} \leq d_{u0} + d_{0v}$ for $uv \in E(C^*)$ by the triangular inequality, the definition that $d_{i0} = d_{0i} = d_i$ for every $i \in V$, the fact that each vertex i is an extreme of exactly two edges of C^*, and the definition $d_0 = 0$. ■

Theorem 6. *There exists a factor $4 - \frac{4}{3s^3Q^2}$ polynomial-time approximation algorithm for the cumulative VRP.*

Proof. Given as input an instance of cumulative VRP as previously described, consider the algorithm with the steps: (1) compute a tour C of G by the well-known Christofides' algorithm [Gaur et al. 2013], which guarantees that $|C| \leq \frac{3}{2}|C^*|$; (2) compute a cluster partition P^* of tour C with optimal fuel consumption by a DP algorithm in time $O(n^2)$ as done in [Gaur et al. 2013]; and (3) return the subtours $C_1^*, C_2^*, \ldots, C_k^*$ of P^*.

In this algorithm, we optimally calculate P^* of C in polynomial time. The analysis is being made over a heuristic algorithm that also calculates a cluster partition P of a tour, thus we can state that $f(P^*) \leq f(P)$. Let S^* be an optimal routing scheduling in fuel consumption. By theorems 5 and 3, we have the ratio

$$\frac{f(P^*)}{f(S^*)} \leq \frac{\left(1 + \frac{2}{\beta}\right) b \left(\sum_{i=1}^{n} w_i d_i\right) + \left(1 + \frac{\beta}{2}\right) a|C| + 4a \frac{\sum_{i=1}^{n} w_i d_i}{Q} - 2a \frac{\sum_{j=1}^{k} |C_j|}{sQ}}{a \max\left(|C^*|, 2 \frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + b \sum_{i=1}^{n} w_i d_i}$$

$$\leq \frac{4\left(a\left(\frac{|C^*|}{2} + \frac{2}{2}\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + b\sum_{i=1}^{n} w_i d_i\right) - 2a\frac{\sum_{j=1}^{k}|C_j|}{sQ}}{a\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + b\sum_{i=1}^{n} w_i d_i} \tag{1}$$

$$\leq 4 - \frac{2a\sum_{j=1}^{k}|C_j|}{sQ\left(a\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + b\sum_{i=1}^{n} w_i d_i\right)}$$

$$\leq 4 - \frac{2a\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right)}{sQ\left(a\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + b\sum_{i=1}^{n} w_i d_i\right)} \tag{2}$$

$$= 4 - \frac{2\mu W_0\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right)}{sQ\left(\mu W_0\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + \mu\sum_{i=1}^{n} w_i d_i\right)} \tag{3}$$

$$\leq 4 - \frac{2W_0\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right)}{sQ\left(sW_0\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + sW_0\sum_{i=1}^{n} sw_i d_i\right)} \tag{4}$$

$$= 4 - \frac{2\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right)}{s^2Q\left(\max\left(|C^*|, 2\frac{\sum_{i=1}^{n} w_i d_i}{Q}\right) + \sum_{i=1}^{n} sw_i d_i\right)}$$

$$\leq 4 - \frac{4\frac{\sum_{i=1}^{n} w_i d_i}{Q}}{3s^3Q\sum_{i=1}^{n} w_i d_i} = 4 - \frac{4}{3s^3Q^2}. \tag{5}$$

To obtain (1), we chose $\beta = \frac{2}{3}$ and used that $|C| \leq \frac{3}{2}|C^*|$. We used Theorem 1 to get (2). To obtain (3), recall that $a = \mu W_0$ and $b = \mu$. In (4), we made use of the fact that s and sW_0 are integers. To obtain (5), we deal with the absolute value of the fraction: we kept or lower the numerator; and we majored the denominator applying the fact that $|C^*| \leq 2\sum_{i=1}^{n} d_i \leq 2\sum_{i=1}^{n} sw_i d_i$ as Lemma 1 states and as sw_i are integers for every $i \in V$, and the fact $2\frac{\sum_{i=1}^{n} w_i d_i}{Q} = 2\frac{\sum_{i=1}^{n} sw_i d_i}{sQ} \leq 2\sum_{i=1}^{n} sw_i d_i$, as sQ is integer. ■

References

Altinkemer, K. and Gavish, B. (1987). Heuristics for unequal weight delivery problems with a fixed error guarantee. *Operations Research Letters*, 6(4):149 – 158.

Gaur, D. R., Mudgal, A., and Singh, R. R. (2013). Routing vehicles to minimize fuel consumption. *Operations Research Letters*, 41(6):576 – 580.

Haimovich, M. and Rinnooy Kan, A. H. G. (1985). Bounds and heuristics for capacitated routing problems. *Mathematics of Operations Research*, 10(4):527–542.

Kara, I., Kara, B. Y., and Yetiş, M. K. (2008). *Cumulative Vehicle Routing Problems*, pages 85–98. InTech Education and Publishing KG, Vienna, Austria.