Tight bounds for gap-labellings

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Abstract. *An ordered pair (π, c_π) is said to be a gap-$[k]$-edge-labelling (gap-$[k]$-vertex-labelling) if π is an edge-labelling (vertex-labelling) on the set $\{1,\ldots,k\}$, and c_π is a proper vertex-colouring induced by a gap-function based on π. Gap-$[k]$-edge-labellings and gap-$[k]$-vertex-labellings were first introduced by M. Tahraoui et al. [7] and A. Dehghan et al. [2], respectively. The edge-gap number (vertex-gap number) is the least k for which there exists a gap-$[k]$-edge-labelling (gap-$[k]$-vertex-labelling) of a graph. In this work, we study the edge-gap number, χ^g_E, and the vertex-gap number, χ^g_V, of cycles, crowns and wheels.*

1. Introduction

Let G be a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. An edge $e \in E(G)$ with ends $u, v \in V(G)$ is denoted by uv. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$ and the minimum degree of G, by $\delta(G)$. The set of edges incident with v is denoted by $E(v)$ and its neighbourhood, by $N(v)$.

For a set $\mathcal{C}$ of *colours*, a *(proper vertex-)colouring* of G is a mapping $c : V(G) \to \mathcal{C}$, such that $c(u) \neq c(v)$ for every pair of adjacent vertices $u, v \in V(G)$. If $|\mathcal{C}| = k$, mapping c is called a *k-colouring*. The *chromatic number*, $\chi(G)$, is the least number k for which G admits a k-colouring. For $S = E(G)$ or $S = V(G)$ and a set of *labels* $[k] = \{1,\ldots,k\}$, a *labelling* π of G is a mapping $\pi : S \to [k]$. For $S' \subseteq S$, the *gap function*, gap(π, S'), is defined as: 1, if $S' = \emptyset$; $\pi(s)$, if $S' = \{s\}$; or $\max_{s\in S}\{\pi(s)\} - \min_{s\in S}\{\pi(s)\}$, if $|S| \geq 2$. A *gap-$[k]$-edge-labelling* of G is an ordered pair (π, c_π) such that $\pi : E(G) \to [k]$ is a labelling of G and $c_\pi : V(G) \to \mathcal{C}$, a colouring of G defined as $c_\pi(v) = \text{gap}(\pi, E(v))$. The least k for which G admits a gap-$[k]$-edge-labelling, $\chi^g_E(G)$, is called *edge-gap number*. A *gap-$[k]$-vertex-labelling* of G is defined similarly, with $\pi : V(G) \to [k]$, and $c_\pi(v) = \text{gap}(\pi, N(v))$. The least k for which G admits a gap-$[k]$-vertex-labelling, $\chi^g_V(G)$, is called *vertex-gap number*. An interesting remark is that all K_2-free graphs admit a gap-$[k]$-edge-labelling for some k, while there are graphs for which there is no gap-$[k]$-vertex-labelling, for any k [7]. For instance, complete graphs K_n, $n \geq 4$, do not admit such a labelling.

Most researchers date the labelling of graphs using mathematical operations back to 1967, when it was introduced by A. Rosa [3, 4]. Since then, several variants of labellings have been created and studied. Gap-$[k]$-edge-labellings, introduced in 2012 by M. Tahraoui et al. [7] as a variant of gap-k-colourings, were investigated by R. Scheidweiler and E. Triesch [5, 6], and A. Brandt et al. [1]. The latter proved that $\chi(G) \leq \chi^g_E(G) \leq \chi(G)+1$ for all graphs except stars. They also determined the edge-gap number for complete graphs, cycles and trees. The vertex variant was first introduced in 2013 by A. Dehghan et al. [2], who proved that deciding whether a graph admits a gap-$[k]$-vertex-labelling is NP-complete for several classes of graphs. These findings inspired us to further research the properties of these labellings. In this work, we study the edge-gap and vertex-gap numbers for three classes of graphs: cycles, crowns and wheels, and compare these parameters.

2. Results

We start by considering cycle graphs C_n, which are 2-regular, connected, simple graphs. Let $V(C_n) = \{v_0, \ldots, v_{n-1}\}$ and $E(C_n) = \{v_i v_{i+1}\}$, $0 \leq i < n$. For C_3, $\chi_V^g(C_3) = \chi_E^g(C_3) = 4$. For $n \geq 4$, the vertex-gap number is established in the next theorem.

Theorem 1. *Let $G \cong C_n$, $n \geq 4$. Then, $\chi_V^g(G) = 2$, if $n \equiv 0 \pmod 4$, and $\chi_V^g(G) = 3$, otherwise.*

Outline of the proof. Let $G = C_n$, $n \geq 4$. Since $\delta(G) \geq 2$, $\chi_V^g(G) \geq \chi(G)$. It is well-known that $\chi(G) = 2$ if n is even, and $\chi(G) = 3$, otherwise. By induction on n, we show that $\chi_V^g(G) \leq 3$.

We prove that G admits a gap-[3]-vertex-labelling (π, c_π) with labels $(\pi(v_{n-2}), \pi(v_{n-1}), \pi(v_0))$ and colours $(c_\pi(v_{n-2}), c_\pi(v_{n-1}), c_\pi(v_0))$ satisfying one of the following conditions: (i) (1, 2, 1) and (1, 0 , 1); (ii) (2, 3, 2) and (2, 0, 2); (iii) (3, 1, 3) and (1, 0, 1); or (iv) (1, 1, 1) and (2, 0, 2).

For C_4 and C_5, assign labels $(1, 3, 1, 2)$ and $(1, 3, 1, 1, 2)$ to vertices $(v_0, \ldots, v_{n-1})$, respectively, satisfying condition (i). Now, let (π, c_π) be a gap-[3]-vertex-labelling for C_n, $n \geq 4$, satisfying one of the above conditions. We create cycle C_{n+2} by replacing vertex v_{n-1} with a P_3, and labelling the new vertices so that if C_n satisfies condition l, $l \in \{\text{(i)}, \text{(ii)}, \text{(iii)}, \text{(iv)}\}$, then C_{n+2} satisfies the next condition in the cyclic order $(\text{(i)}, \text{(ii)}, \text{(iii)}, \text{(iv)})$. We remark that all operations on the indices are taken modulo n. Figure 1 exemplifies this construction.

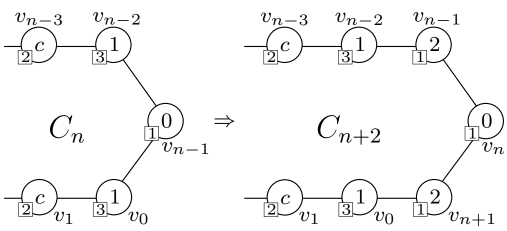

Figure 1. Cycle C_n satisfies condition (iii). We assign labels to the vertices of C_{n+2} so as to satisfy condition (iv)

Now, we prove that only C_n, $n \equiv 0 \pmod 4$ admits a gap-[2]-vertex-labelling. Let (π, c_π) be a gap-[2]-vertex-labelling of C_n. Adjust notation so that $c_\pi(v_i) = 0$, if $i \equiv 0 \pmod 2$, and $c_\pi(v_i) = 1$, otherwise. This implies that every vertex v_i with odd index has the same label a, for $a \in \{1, 2\}$, and also $\{\pi(v_{i-1}), \pi(v_{i+1})\} = \{1, 2\}$. Let $j \equiv 1 \pmod 2$. Each sequence of four vertices $(v_{j-1}, v_j, v_{j+1}, v_{j+2})$ has labels (a, a, b, a) or (b, a, a, a), for $\{a, b\} = \{1, 2\}$. Moreover, the distance between any two consecutive vertices $u, v \in V(C_n)$ with label b is exactly four. Without loss of generality, consider the sequence (a, a, b, a) starting at v_0 and repeating itself along the cycle. If $n \equiv 2 \pmod 4$, then $c_\pi(v_{n-1}) = c_\pi(v_0) = 0$ and, therefore, c_π is not a colouring of G. Thus, there is no gap-[2]-vertex-labelling of G in this case. For $n \equiv 0 \pmod 4$, any assignment of values $\{a, b\} = \{1, 2\}$ using one of the sequences (a, a, b, a) or (b, a, a, a) produces a gap-[2]-vertex-labelling for C_n, and the result follows. □

In 2016, A. Brandt et al. [1] showed that $\chi_E^g(C_n) = 2$ if $n \equiv 0 \pmod 4$, and $\chi_E^g(C_n) = 3$, otherwise. Their proof is shorter, although with some common points. As we have been observing, gap-$[k]$-edge-labellings allow an analysis in a more restricted neighbourhood, which helps to limit the number of cases under consideration.

Next, we study crown graphs. A *crown* R_n, $n \geq 3$, is the graph obtained from C_n and n copies of K_2, by identifying each vertex of C_n with one vertex of a different copy of a K_2. Figure 2 illustrates the crown R_8. The values of $\chi^{g}_{E}(R_n)$ and $\chi^{g}_{V}(R_n)$ are established in Theorem 2.

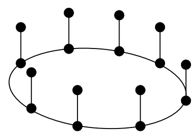

Figure 2. The crown R_8

Theorem 2. *Let $G \cong R_n$, $n \geq 3$. Then, $\chi^{g}_{E}(G) = \chi^{g}_{V}(G) = \chi(C_n)$.*

Proof. Let $G = R_n$, with $V(G) = \{v_0, \ldots, v_{n-1}\} \cup \{u_0, \ldots, u_{n-1}\}$, $d(v_i) = 3$ and $d(u_i) = 1$, $0 \leq i < n$. First, we show that $\chi^{g}_{E}(G) \geq \chi(C_n)$. Note that $\chi^{g}_{E}(G) \geq 2$ since G has adjacent vertices of same degree. However, if n is odd, $\chi^{g}_{E}(G) \geq 3$ since it is not possible to have colour 2 in a vertex of degree three in any gap-[2]-edge-labelling of G.

Let $\pi(v_iv_{i+1}) = 1$, $0 \leq i < n$, and $\pi(v_iu_i) = 1 + (i \bmod 2)$, $0 \leq i < n - (n \bmod 2)$. If n is odd, let $\pi(v_{n-1}u_{n-1}) = 3$. Observe that $c_\pi(u_i) = \pi(v_iu_i)$ for all u_i, $c_\pi(v_i) = i \bmod 2$, for $i \in [0, n - (n \bmod 2)]$, and $c_\pi(v_{n-1}) = 2$, when n is odd. Therefore, (π, c_π) is a gap-$[\chi(C_n)]$-edge-labelling, and we conclude that $\chi^{g}_{E}(G) = \chi(C_n)$.

Now, we prove that $\chi^{g}_{V}(G) = \chi(C_n)$. As in the previous case, $\chi^{g}_{V}(G) \geq \chi(C_n)$. In order to conclude the proof, it is sufficient to construct a gap-$[\chi(C_n)]$-vertex-labelling for G. Let $\pi(v_i) = \chi(R_n)$, $0 \leq i < n$, and $\pi(u_i) = 1 + (i \bmod 2)$, $0 \leq i < n - (n \bmod 2)$. If n is odd, let $\pi(u_{n-1}) = 3$. The result follows from the fact that $c_\pi(u_i) = \pi(v_i) = \chi(R_n)$ for all u_i, $c_\pi(v_i) = \chi(R_n) - 1 - (i \bmod 2)$, and $c_\pi(v_{n-1}) = 0$, when n is odd. □

The last class considered is the wheel graphs. A *wheel* W_n, $n \geq 3$ is the graph obtained from R_n by identifying all degree-one vertices. In the next theorem we determine $\chi^{g}_{E}(W_n)$.

Theorem 3. *Let $G \cong W_n$, $n \geq 3$. Then, $\chi^{g}_{E}(G) = 3$, if n is even and $n \neq 4$, and $\chi^{g}_{E}(G) = 4$, otherwise.*

Proof. Let $G = W_n$, $n \geq 3$. Since $\delta(G) \geq 2$, $\chi^{g}_{E}(G) \geq \chi(G) = \chi(C_n) + 1$. Consider, first, the case where n is even. Assign label 2 to edges v_iv_{i+1}, $0 \leq i < n$. Following the order of the indices, assign labels 2, 1, alternately, to edges v_iv_n, $0 \leq i < n - 1$. Assign label 3 to the remaining edge $v_{n-1}v_n$. Observe that $c_\pi(v_i) = i \bmod 2$, $0 \leq i < n$, and $c_\pi(v_n) = 2$. We conclude that (π, c_π) is a gap-[3]-edge-labelling of G.

Now take $n = 3$. Since $G \cong K_4$, $\chi^{g}_{E}(G) = \chi^{g}_{E}(K_4) = 4$, as shown by A. Brandt et al. [1]. Finally, suppose $n \geq 5$ and odd. Assign labels 2, 3, alternately, to the edges v_iv_{i+1}, $1 \leq i \leq n - 3$; labels 1, 2 to the edges v_iv_n, $0 \leq i \leq n - 3$; and labels 3, 1, 1, 4, 1 to the edges $v_{n-2}v_{n-1}$, $v_{n-1}v_0$, v_0v_1, $v_{n-2}v_n$, $v_{n-1}v_n$, respectively. In order to see that (π, c_π) is a gap-[4]-edge-labelling, note that $c_\pi(v_i) = 2 - (i \bmod 2)$, $1 \leq i \leq n - 1$, $c_\pi(v_0) = 0$ and $c_\pi(v_n) = 3$. This concludes the proof. □

Finally, consider the gap-$[k]$-vertex-labelling of wheels. As previously stated, the graph $W_3 \cong K_4$ does not admit a gap-$[k]$-vertex-labelling for any k. It remains to consider $\chi_V^g(W_n)$ for $n \geq 4$, which is established in the next theorem.

Theorem 4. *Let $G \cong W_n$, $n \geq 4$. Then, $\chi_V^g(G) = 3$, if n is even and $n \geq 8$, and $\chi_V^g(G) = 4$, otherwise.*

Proof. Let $G = W_n$, $n \geq 4$, with $V(W_n) = \{v_0, \ldots, v_n\}$ and $d(v_n) = n - 1$. Since $\delta(G) \geq 2$, $\chi_V^g(G) \geq \chi(G) = \chi(C_n) + 1$. For $n = 4$, assign labels $4, 1, 4, 1, 3$ to vertices $v_0, \ldots, v_4$, respectively, obtaining a gap-$[4]$-vertex-labelling of W_4. Now, for $n \geq 5$, assign label 2 to vertices v_0, v_1, v_2 and v_n. Assign labels 4, 1, alternately, to the remaining vertices v_i, $3 \leq i < n$. Note that $c_\pi(v_i) = 2 - (i \bmod 2)$, $2 \leq i < n$, $c_\pi(v_0) = 2 - (n \bmod 2)$, $c_\pi(v_1) = 0$, and $c_\pi(v_n) = 3$. This is a gap-$[4]$-vertex-labelling of G, and the result follows.

Now, consider the case $n \geq 8$, $n \equiv 0 \pmod 2$. Assign labels 2, 1, alternately, to vertices v_i, $0 \leq i \leq n - 6$. Assign label 3 to vertex v_{n-3}. Assign label 2 to the remaining vertices v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n. Note that $c_\pi(v_i) = 1 + (i \bmod 2)$, $0 \leq i < n$, and $c_\pi(v_n) = 2$. Therefore, (π, c_π) is a gap-$[3]$-vertex-labelling of G.

In order to complete the proof, it remains to consider the cases of $n = 4$ and $n = 6$. Since these are small cases, one can see, by inspection, that there are no gap-$[3]$-vertex-labellings of W_4 or W_6 by considering the possible colours of v_n. □

3. Concluding remarks

In this work, we studied $\chi_E^g(G)$ and $\chi_V^g(G)$ for cycles, crowns and wheels and observed that the edge-labelling variant is less restrictive than the vertex one. This occurs because a labelled edge only affects the colours of its endpoints, whereas a labelled vertex affects its entire neighbourhood. Moreover, for the classes considered in this work, it is possible to assign different labels to a certain edge, maintaining the resulting vertex colouring, whereas such a property is not true for gap-$[k]$-vertex-labellings.

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References

[1] A. Brandt, B. Moran, K. Nepal, F. Pfender, D. Sigler, *Local gap coloring from edge labelings*, Australasian Journal of Combinatorics **65(3)** (2016), 200-211.

[2] A. Dehghan, M.-R. Sadeghi and A. Ahad, *Algorithmic complexity of proper labeling problems*, Theoretical Computer Science **495** (2013), 25 - 36.

[3] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics #DS6 (2016)

[4] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (International Symposium) (1967), 394-355.

[5] R. Scheidweiler, E. Triesch, *New estimates for the gap chromatic number*, SIAM Journal of Discrete Mathematics **328** (2014), 42-53.

[6] R. Scheidweiler, E. Triesch, Gap-neighbour-distinguishing colourings, Journal of Combinatorial Mathematics and Combinatorial Computing, **94** (2015), 205-214.

[7] M. A. Tahraoui, E. Duchene, H. Kheddouci, *Gap vertex-distinguishing edge colourings of graphs*, Discrete Mathematics **312** (20) (2012), 3011 - 3025.